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GRAHAM–ROTHSCHILD PARAMETER WORDS AND MEASURABLE PARTITIONS

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Generalizing earlier results of Moran and Strauss (Mathematika 27, 1980, 213–224) and of Carlson and Simpson (Adv. in Math. 53, 1984, 265–290) it was shown in Prömel and Voigt (Trans. Amer. Math. Soc. 291, 1985, 198–201) that Baire sets of k-parameter words are Ramsey. Motivated by the "duality" between category and measure, we investigate in this paper measurable sets of k parameter words. We show that measurable sets of ascending k-parameter words are Ramsey, whereas in general measurable sets of k-parameter words fail to be Ramsey.

1. Introduction

This paper is concerned with Graham–Rothschild parameter words and related structures. For a general background we refer the reader to Graham, Rothschild and Spencer [4], Prömel and Voigt [11] or Prömel and Voigt [12].

In 1977 R. Graham and B. Rothschild [3] introduced the notion of 'k-parameter sets' and proved a Ramsey type partition theorem for partitions of k-parameter subsets in the n-dimensional cube A^n, where A is a finite set. The case $k = 0$ turns out to be the celebrated Hales–Jewett theorem [5]. For about 15 years it was not clear how to extend the Graham-Rothschild theorem to partitions of the infinite dimensional cube A^ω. One obstacle is that the axiom of choice prevents straightforward generalizations. However, already in 1978 G. Moran and D. Strauss [9], not being aware of Hales–Jewett's theorem, established infinite versions of Hales–Jewett's theorem. In the Moran–Strauss paper two approaches are suggested. The first is a *topological approach*. It is shown that Baire partitions $A^\omega \to \omega$ are Ramsey. The second is a *measure-theoretic approach*. It is shown that measurable partitions $A^\omega \to \omega$ are Ramsey. Unfortunately, the Moran-Strauss paper was overlooked and did not get the attention it should have deserved. The major breakthrough was, then, achieved by T. Carlson and S. Simpson [2] who showed that Borel sets of k-parameter words are Ramsey. This has been generalized by the authors to *Baire sets of k-parameter words are Ramsey* (Prömel and Voigt [10]). This is in some sense best possible, as S. Shelah [13] shows that the consistency of *ZFC* implies the consistency of *ZF* plus the axiom of dependent choices plus *every set of k-parameter words is Baire*.

It is a natural question whether the *duality* between category and measure extends to k-parameter words. As we will show (Theorem C) this is in general

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not the case, viz., there exists measurable sets of k-parameter words which are not Ramsey.

Our counterexample is based on the fact that there exist sets of k-parameter words of measure zero which are not Ramsey null i.e., which cannot be avoided. A construction of such sets uses that, in general, each parameter occurs infinitely often. If we restrict our consideration to so-called ascending k-parameter words, i.e., infinite parameter-word where each parameter occurs only finitely often and the last occurrence of the ith parameter comes before the first occurrence of the $i+1$st parameter for every i, then it can be proved (Theorem D) that measurable sets of such ascending k-parameter words are Ramsey. Ascending parameter words have been studied earlier by K. Milliken [8] and T. Carlson [1]. The topological counterpart to Theorem D, i.e., that Baire sets of ascending k-parameter words are Ramsey was proved in Prömel, Voigt [10].

2. Definitions and known results

Conventions: small Latin letters k, l, m, n, ... denote nonnegative integers; we write $[k] = \{0, 1, \ldots, k-1\}$ for the set of predecessors of k; $\omega = \{0, 1, 2, \ldots\}$ is the smallest infinite ordinal; small Latin letters f, g, h denote finite sequences (resp. finite parameter words); capital letters F, G, H denote infinite sequences (resp., infinite parameter words); $\otimes$ denotes the concatenation of sequences.

Definition. Let A be a finite set, $A \cap \omega = \emptyset$. Let $\alpha \leq \beta \leq \omega$. $[A]\binom{\alpha}{\beta}$ is the set of all sequences (mappings) $F : [\beta] \to A \cup [\alpha]$ such that

(1) for every $j < \alpha$ there exists $i < \beta$ with $F(i) = j$,

(2) $\min F^{-1}(i) \leq \min F^{-1}(j)$ for all $i < j < \alpha$.

The elements $F \in [A]\binom{\beta}{\alpha}$ are *α-parameter words of length β over alphabet A*. The values $i < \alpha$ are interpreted as *parameters*, whereas elements $a \in A$ are *constants*. Notice that each parameter has to appear at least once. For $F \in [A]\binom{\gamma}{\beta}$ and $G \in [A]\binom{\beta}{\alpha}$ the composite $F \cdot G \in [A]\binom{\gamma}{\alpha}$ is defined as the insertion of G into the parameters of F, i.e., $(F \cdot G)(i) = G(F(i))$ if $F(i) \in [\alpha]$ and $(F \cdot G)(i) = F(i)$ otherwise.

We also define $[A]_{asc}\binom{\beta}{\alpha}$, the set of *ascending* α-parameter words of length β over alphabet A, by

$$[A]_{asc}\binom{\beta}{\alpha} = \left\{ F \in [A]\binom{\beta}{\alpha} \mid F^{-1} \text{ is finite and } \max F^{-1}(i) < \min F^{-1}(j) \text{ for all } i < j < \alpha \right\}.$$

One easily observes that ascending parameter words are closed under the composition defined above.

Remark. The Graham-Rothschild partition theorem [3] can be formulated saying that for every partition $[A]\binom{n}{k} = \mathcal{A}_0 \cup \ldots \cup \mathcal{A}_{r-1}$, where $n \geq n(A, m, k, r)$ is sufficiently large, there exist $f \in [A]\binom{n}{m}$ and $j < r$ such that $f \cdot [A]\binom{m}{k} \subseteq \mathcal{A}_j$. The particular case $k = 0$, dealing with partitions of A^n, is Hales–Jewett's theorem.

Concepts from Topology. The set $[A]\binom{\omega}{k}$ naturally becomes a metric space defining $d(F, G) = 1/(n+1)$ if and only if $n = \{l < \omega \mid f(l) \neq G(l)\}$. Note that $[A]\binom{\omega}{k} \subseteq (A \cup [k])^\omega$ and the topology is just the Tychonoff product topology with $A \cup [k]$ being discrete. As a matter of fact, $[A]\binom{\omega}{k}$ is a Polish space. It turns out that for every partition $[A]\binom{\omega}{k} = \mathcal{A}_0 \cup \ldots \cup \mathcal{A}_{r-1}$ into sets having the property of Baire (meaning that $\mathcal{A}$ is open modulo some meager set) there exists $F \in [A]\binom{\omega}{\omega}$ and there exists $j < r$ such that $F \cdot [A]\binom{\omega}{k} \subseteq \mathcal{A}_j$; cf. Prömel and Voigt [10], resp., Moran and Strauss [9] for $k = 0$. With respect to partitions into Borel sets this is a result of Carlson and Simpson [2].

Using a different notation, ascending parameter words have been studied by K. Milliken [8] and T. Carlson [1], cf. also Voigt [14]. With respect to $|A| = 1$ the first interesting case appears for $k = 1$. The assertion that for every mapping $\Delta : [\{a\}]_{asc}\binom{\omega}{1} \to [2]$ there exists an $F \in [\{a\}]_{asc}\binom{\omega}{\omega}$ such that $\Delta(F \cdot G) = \Delta(F \cdot H)$ for all $G, H \in [\{a\}]_{asc}\binom{\omega}{1}$ is a reformulation of Hindman's theorem (Hindman [6]). This has been generalized by Milliken [8] to partitions $\Delta : [\{a\}]_{asc}\binom{\omega}{k} \to [2]$ for arbitrary $k < \omega$. Again, due to the axiom of choice, such a result cannot hold for alphabets with more than one element.

As a subset of $[A]\binom{\omega}{k}$ the set $[A]_{asc}\binom{\omega}{k}$ is also a metric space. Notice that $[\{a\}]_{asc}\binom{\omega}{k}$ is a countable space.

Prömel and Voigt [10] have shown that for every partition $[A]_{asc}\binom{\omega}{k} = \mathcal{A}_0 \cup \ldots \cup \mathcal{A}_{r-1}$ into sets having the property of Baire there exists an $F \in [A]_{asc}\binom{\omega}{\omega}$ and there exists $j < r$ such that $F \cdot [A]_{asc}\binom{\omega}{k} \subseteq \mathcal{A}_j$.

As a matter fact, the basic result is that meager sets are Ramsey null:

Theorem A. [10] *Let $\mathcal{M} \subseteq [A]\binom{\omega}{k}$, resp. $\mathcal{M} \subseteq [A]_{asc}\binom{\omega}{k}$ be meager sets. Then there exists an $F \in [A]\binom{\omega}{\omega}$, resp., $F \in [A]_{asc}\binom{\omega}{\omega}$ such that $F \cdot [A]\binom{\omega}{k} \cap \mathcal{M} = \emptyset$, resp. $F \cdot [A]_{asc}\binom{\omega}{k} \cap \mathcal{M} = \emptyset$.*

Concepts from measure. Let μ be completion of the product measure on $[A]\binom{\omega}{k}$ which is generated by some measure $\tilde{\mu}$ on $A \cup [k]$ such that $\tilde{\mu}(x) > 0$ for every $x \in A \cup [k]$. This induces a measure on the Tychonoff cones by $\mu\Big(\mathcal{T}((g_0, \ldots, g_{n-1}), n)\Big) = \prod_{i<n} \tilde{\mu}(g_i)$. By Caratheodory's theorem this may be extended (uniquely) to a measure which is defined on the Borel sets. We take the completion by adding all subsets of sets of measure zero and call the such generated σ-algebra the σ-algebra of *measurable sets*. For $k = 0$ G. Moran and D. Strauss [9] show that for every partition $A^\omega = \mathcal{A}_0 \cup \ldots \cup \mathcal{A}_{r-1}$ into measurable sets there exists $F \in [A]\binom{\omega}{\omega}$ and there exists $j < r$ such that $F \cdot [A]^\omega \subseteq \mathcal{A}_j$. They even prove a density result:

Theorem B. [9] *For every set $\mathcal{M} \subseteq [A]\binom{\omega}{0}$ of positive measure there exists $F \in [A]\binom{\omega}{\omega}$ such that $F \cdot [A]\binom{\omega}{0} \subseteq \mathcal{M}$.*

Such a density result does not hold for $k > 0$, as it can be seen from the set $\mathcal{M} = \{G \in [A]\binom{\omega}{k} \mid G(0) = 0\}$. $\mathcal{M}$ consists of all k-parameter words which start with a parameter. $\mathcal{M}$ has positive measure, but for every $F \in [A]\binom{\omega}{\omega}$ there exists $G \in [A]\binom{\omega}{k}$ with $F \cdot G \notin \mathcal{M}$. Simply let G be such that $G(0) \in A$.

Notation. For $g \in A^n$ and $\mathcal{D} \subseteq A^\omega$ the expression $g \otimes A \otimes \mathcal{D}$ stands for the set $\{g \otimes (a) \otimes H \mid a \in A,\ H \in \mathcal{D}\}$.

We shall need the following lemma, which implicitly is already contained in Moran and Strauss [9]:

Lemma 1. *Let $\mathcal{C} \subseteq A^\omega$ be a set of positive measure. Then there exists a finite sequence $g = (g_0, \dots, g_{n-1}) \in A^n$, for some $n < \omega$, and there exists a set $\mathcal{D} \subseteq A^\omega$ of positive measure such that $g \otimes A \otimes \mathcal{D} \subseteq \mathcal{C}$.*

Proof. Without loss of generality we can assume that $\mathcal{C}$ is closed (otherwise there exists $\mathcal{C}' \subseteq \mathcal{C}$, $\mathcal{C}'$ closed and $\mu(\mathcal{C}') > \mu(\mathcal{C})/2$. Let $\epsilon > 0$ be such that $\epsilon \cdot \sum_{x \in A} \mu(x)^{-1} < 1$. Let $\mathcal{W} \subseteq A^\omega$ be an open set such that $\mathcal{C} \subseteq \mathcal{W}$ and $\mu(\mathcal{W}) < \mu(\mathcal{C}) \cdot (1 + \epsilon)$.

For $g \in A^n$ let $\mathcal{T}(g, n) = \{F \in A^\omega \mid F\rceil[n] = g\}$ be the *Tychonoff cone* generated by g. The set of all Tychonoff cones forms a basis for the topology an A^ω. So $\mathcal{W}$ can be written as a union of Tychonoff cones. As $\mathcal{C}$ is compact there exist finitely many Tychonoff cones $\mathcal{T}(g_i, n_i)$, $i = 0, \dots, m-1$, such that $\mathcal{C} \subseteq \bigcup_{i<m} \mathcal{T}(g_i, n_i)$. As we may assume that $\mathcal{T}(g_i, n_i) \cap \mathcal{T}(g_j, n_j) = \emptyset$ for $i \neq j$ and thus $\mu\left(\bigcup_{i<m} \mathcal{T}(g_i, n_i)\right) = \sum_{i<m} \mu(\mathcal{T}(g_i, n_i)) < \mu(\mathcal{C}) \cdot (1+\epsilon)$, there exists $i < m$ such that $\mu(\mathcal{T}(g_i, n_i)) < \mu(\mathcal{C} \cap \mathcal{T}(g_i, n_i)) \cdot (1 + \epsilon)$.

Let $\mathcal{C}' \subseteq A^\omega$ be such that $\mathcal{C} \cap \mathcal{T}(g_i, n_i) = g_i \otimes \mathcal{C}'$. Then $\mathcal{C}'$ is measurable and $\mu(\mathcal{C}') = \dfrac{\mu(\mathcal{C} \cap \mathcal{T}(g_i, n_i))}{\mu(\mathcal{T}(g_i, n_i))} > \dfrac{1}{1+\epsilon} > 1 - \epsilon$. Thus there exists $\mathcal{D} \subseteq A^\omega$ with $\mu(\mathcal{D}) > 1 - \epsilon \cdot \sum_{x \in A} \mu(x)^{-1} > 0$ such that $A \otimes \mathcal{D} \subseteq \mathcal{C}'$. Altogether we obtain $\mathcal{D} \subseteq A^\omega$ with $\mu(\mathcal{D}) > 0$ and $g = g_i \in A^{n_i}$ such that $g \otimes A \otimes \mathcal{D} \subseteq \mathcal{C}$, as desired. ∎

3. Measurable sets in $[A]\binom{\omega}{k}$

As in the preceding section let μ be the completion of the product measure $\tilde{\mu}$ on $[A]\binom{\omega}{k}$ which is generated by some nontrivial measure on $A \cup [k]$.

From Theorem A it follows that meager sets in $[A]\binom{\omega}{k}$ are Ramsey null. The Moran–Strauss result Theorem B implies that, with respect to $k = 0$, sets $\mathcal{N} \subseteq A^\omega$ of measure zero are Ramsey null. It turns out that the duality between category and measure does not extend to $k > 0$:

Theorem C. *Let A be a finite nonempty set and let $k > 0$. Then there exists a set $\mathcal{N} \subseteq [A]\binom{\omega}{k}$ of measure zero such that for every $F \in [A]\binom{\omega}{\omega}$ there exists $G \in [A]\binom{\omega}{\omega}$ such that $F \cdot G \cdot [A]\binom{\omega}{k} \subseteq \mathcal{N}$.*

In particular, this result implies that, in general, measurable sets in $[A]\binom{\omega}{k}$ fail to be Ramsey:

Corollary. *Let A and k be as above. There exists a partition $[A]\binom{\omega}{k} = \mathcal{A}_0 \cup \mathcal{A}_1$ into measurable sets such that for all $F \in [A]\binom{\omega}{\omega}$ there exist G, $H \in [A]\binom{\omega}{k}$ such that $F \cdot G \in \mathcal{A}_0$ and $F \cdot H \in \mathcal{A}_1$.*

Proof of Corollary. Let $\mathcal{I} \subseteq [A]\binom{\omega}{k}$ be a Bernstein set, i.e., a set such that every nonempty perfect (closed without isolated points) subset $\mathcal{P} \subseteq [A]\binom{\omega}{k}$ intersects $\mathcal{I}$ as well as its complement $[A]\binom{\omega}{k} \setminus \mathcal{I}$. Using the axiom of choice such a Bernstein set may be constructed straightforwardly, cf. Kuratowski [7]. Let $\mathcal{N} \subseteq [A]\binom{\omega}{k}$ be according to Theorem C. Define the partition by $\mathcal{A}_0 = \mathcal{N} \cap \mathcal{I}$ and $\mathcal{A}_1 = [A]\binom{\omega}{k} \setminus \mathcal{A}_0$. Notice that, although the Bernstein set $\mathcal{I}$ itself is not measurable, the intersection $\mathcal{N} \cap \mathcal{I}$ is measurable. The reason is that $\mathcal{N}$ has measure zero. In view of Theorem C we may restrict attention to those $F \in [A]\binom{\omega}{\omega}$ for which $F \cdot [A]\binom{\omega}{k} \subseteq \mathcal{N}$.

For every $F \in [A]\binom{\omega}{\omega}$ the set $F \cdot [A]\binom{\omega}{k}$ contains a nonempty perfect set. E.g., $F \cdot \{G \in [A]\binom{\omega}{k} \mid G\rceil[k] = (0, 1, \dots, k-1)\}$ is perfect. In fact, $F \cdot [A]\binom{\omega}{k}$ is the union of countably many perfect sets. Hence, $F \cdot [A]\binom{\omega}{k}$ intersects $\mathcal{A}_0$ as well as $\mathcal{A}_1$. ∎

Proof of Theorem C. Let

$$\mathcal{M} = \left\{ H \in [A]\binom{\omega}{k} \;\middle|\; \lim_{n\to\infty} \frac{|\{i < n \mid H(i) \in A\}|}{n} = \frac{|A|}{|A|+k} \right\}.$$

From the strong law of large numbers it follows that $\mathcal{M}$ has measure one. Hence the complementary set $\mathcal{N} = [A]\binom{\omega}{k} \setminus \mathcal{M}$ has measure zero. We claim that $\mathcal{N}$ has the desired properties. So let $F \in [A]\binom{\omega}{\omega}$ be arbitrary. We may assume that there exists some $H \in [A]\binom{\omega}{k}$ such that $H^{-1}(j)$ is infinite for every $j < k$ and such that $F \cdot H \in \mathcal{M}$. Otherwise we are done straightforwardly, e.g. let $G \in [A]\binom{\omega}{\omega}$ be such that $G^{-1}(j)$ is infinite for every $j < \omega$, then $F \cdot G \in [A]\binom{\omega}{\omega}$ be such that $F \cdot G \cdot [A]\binom{\omega}{k} \subseteq \mathcal{N}$. By choice of H it follows that $\lim_{n\to\infty} \frac{|\{i < n \mid (F \cdot H)(i) \notin A\}|}{n} = \frac{k}{[A]+k}$. Thus, there exists some $j < k$ such that $\limsup_{n\to\infty} \frac{|\{i < n \mid (F \cdot H)(i) = j\}|}{n} < \frac{k}{|A|+k}$. Then any $G \in [A]\binom{\omega}{\omega}$ such that $G(i) \in A$ whenever $H(i) \neq j$ has the property that $\liminf_{n\to\infty} \frac{|\{i < n \mid (F \cdot G)(i) \in A\}|}{n} > \frac{|A|}{|A|+k}$ and thus $F \cdot G \cdot [A]\binom{\omega}{k} \subseteq \mathcal{N}$, as desired. ∎

4. Measurable sets in $[A]_{asc}\binom{\omega}{k}$ are Ramsey

In this section we shall see that with respect to ascending parameter words the situation is slightly different. Here the dualization of Theorem A holds.

Notice that, as a subset of $[A]\binom{\omega}{k}$, the set of ascending k-parameter words is a closed set of measure zero. Hence a few comments on the measure of $[A]_{asc}\binom{\omega}{k}$ might be helpful: $[A]\binom{\omega}{k}$ may be partitioned into countably many sets as follows:

$$[A]_{asc}\binom{\omega}{k} = \bigcup_{n<\omega} \bigcup_{\substack{f \in [A]_{asc}\binom{n}{k} \\ f(n-1)=k-1}} f \otimes A^{\omega}\,.$$

For each $f \in [A]_{asc}\binom{n}{k}$ with $f(n-1) = k-1$ let $f \otimes A^\omega$ be endowed with the completion of the product measure on A^ω which is generated by some nontrivial measure on A, let us denote this completion by μ_f. Also let

$$\nu : \left\{ f \in [A]_{asc}\binom{n}{k} \;\middle|\; n < \omega,\ f(n-1) = k-1 \right\} \longrightarrow]0,1]$$

be a valuation such that $\sum \nu(f) = 1$. A set $\mathcal{A} \subseteq [A]_{asc}\binom{\omega}{k}$ is measurable if each intersection $\mathcal{A} \cap g \otimes A^\omega$ is measurable.

The measure $\mu(\mathcal{A})$ is the weighted sum of the measures $\mu_f(\mathcal{A} \cap f \otimes A^\omega)$, i.e.,

$$\mu(\mathcal{A}) = \sum_{n<\omega} \sum_{\substack{f \in [A]_{asc}\binom{n}{k} \\ f(n-1)=k-1}} \nu(f) \cdot \mu_f(\mathcal{A} \cap f \otimes A^\omega).$$

The first result is that sets of measure zero in $[A]_{asc}\binom{\omega}{k}$ are Ramsey null.

Theorem D. *Let $\mathcal{N} \subseteq [A]_{asc}\binom{\omega}{k}$ be a set of measure zero. Then there exists an $F \in [A]_{asc}\binom{\omega}{\omega}$ such that $F \cdot [A]_{asc}\binom{\omega}{k} \cap \mathcal{N} = \emptyset$.*

As a corollary we obtain that measurable sets in $[A]_{asc}\binom{\omega}{k}$ are Ramsey:

Corollary. *Let $[A]_{asc}\binom{\omega}{k} = \mathcal{A}_0 \cup \ldots \cup \mathcal{A}_{r-1}$ be a partition into measurable sets. Then there exists an $F \in [A]_{asc}\binom{\omega}{\omega}$ and there exists a $j < r$ such that $F \cdot [A]_{asc}\binom{\omega}{k} \subseteq \mathcal{A}_j$.*

Proof of Corollary. The crucial property we exploit is that every measurable set may be written as a symmetric difference of a Borel set and a set of measure zero. So let $\mathcal{A}_j = \mathcal{B}_j \setminus \mathcal{N}_j \cup \mathcal{N}_j \setminus \mathcal{B}_j$, the $\mathcal{B}_j$ is Borel and $\mu(\mathcal{N}_j) = 0$. Let $\mathcal{N} = \bigcup_{j<r} \mathcal{N}_j$. By Theorem D there exists an $F \in [A]_{asc}\binom{\omega}{\omega}$ such that $F \cdot [A]_{asc}\binom{\omega}{k} \cap \mathcal{N} = \emptyset$.

But then $F \cdot [A]_{asc}\binom{\omega}{k} = \bigcup_{j<r} \mathcal{B}_j \cap F \cdot [A]_{asc}\binom{\omega}{k}$, where each $\mathcal{B}_j \cap F \cdot [A]\binom{\omega}{k}$ is a Borel set. So we may apply Theorem E of Prömel and Voigt [10] which states that Baire sets (in particular, Borel sets) are Ramsey. According to this result, then, there exists some $G \in [A]_{asc}\binom{\omega}{\omega}$ and there exists some $j < r$ such that $F \cdot G \cdot [A]_{asc}\binom{\omega}{k} \subseteq \mathcal{B}_j \cap F \cdot [A]_{asc}\binom{\omega}{k} \subseteq \mathcal{A}_j$. ∎

So it remains to prove Theorem D.

Proof of Theorem D. Let f_0 be the unique element in $[A]_{asc}\binom{k-1}{k-1}$ and let $\mathcal{D}_0 = A^\omega$. Assume by induction that an $f_m \in [A]_{asc}\binom{n_m}{k-1+m}$ and a closed set $\mathcal{D}_m \subseteq A^\omega$ of positive measure have been constructed such that for all $g \in [A]_{asc}\binom{k-1+m}{k}$ it follows that

$$\Big((f_m \cdot g) \otimes A \otimes \mathcal{D}_m\Big) \cap \mathcal{N} = \emptyset\,.$$

For $m = 0$ the requirement holds vacuously. Let $f_m^+ = f_m \otimes (k+m-1)$, we add a new parameter at the end of f_m. For $g \in [A]_{asc}\binom{k+m}{k}$ such that the last position in g is a parameter let $\mathcal{N}_{f_m^+ \cdot g} = \{H \in A^\omega \mid (f_m^+ \cdot g) \otimes H \in \mathcal{N}\}$. Then $\mathcal{N}_m = \bigcup_{\substack{g \in [A]_{asc}\binom{k+m}{k} \\ g(k+m-1)=k-1}} \mathcal{N}_{f_m^+ \cdot g}$ has measure zero in A^ω. Thus, by Lemma 1, there exist an $n < \omega$, and $h \in A^n$ and a closed set $\mathcal{D}_{m+1} \subseteq \mathcal{D}_m \setminus \mathcal{N}_m$ of positive measure such that $f_{m+1} = f_m^+ \otimes h$ again satisfies the inductive requirements. Finally, the limit $F = \bigcap_{m<\omega} \mathcal{I}(f_m, n_m)$ has the desired properties. ∎

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